Calculating monad transformers with category theory

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Abstract

We show that state, reader, writer, and error monad transformers are instances of one general categorical construction: translation of a monad along an adjunction.

1 Introduction

This note is an elaboration of [2] (hence the title). The latter serves as a very gentle introduction to category theory for Haskell programmers. In particular, it explains how monads arise from adjunctions between categories. We take this (arguably well-known in the category theory community) idea one step further. We show that monads can be translated along adjunctions and illustrate by examples how the standard monad transformers — state, reader, writer, error — can be interpreted as instances of this construction. Notably, continuation monad transformers do not fit into this framework.

Unlike [2], this note gives more details, and is as a consequence more technical. It makes greater emphasis on category theory rather than on programming. Like in [2], the reader should consider unproven or partially proven statements to be exercises.

2 Translating monads along adjunctions

We begin by recalling one of the fundamental notions of category theory: the notion of adjunction. The reader is referred to [2] and [3, Chapter IV] for more details.

2.1 Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An *adjunction* from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, U, φ), where $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$ are functors, and φ is a family of bijections

$$\varphi_{X,Y} : \mathcal{D}(FX, Y) \xrightarrow{\sim} \mathcal{C}(X, UY), \quad X \in \operatorname{Ob}\mathcal{C}, \quad Y \in \operatorname{Ob}\mathcal{D},$$

natural in X and Y. We say that F is *left adjoint* to U and U is *right adjoint* to F.

An adjunction $(F, U, \varphi) : \mathcal{C} \to \mathcal{D}$ determines two natural transformations. Namely, for a fixed object $X \in \operatorname{Ob}\mathcal{C}$, the family of functions $\varphi_{X,-} = \{\varphi_{X,Y}\}_{Y \in \operatorname{Ob}\mathcal{D}}$ is a natural transformation from the representable functor $\mathcal{D}(FX, -) : \mathcal{D} \to \mathbf{Set}$ to the functor $\mathcal{C}(X, U(-)) : \mathcal{D} \to \mathbf{Set}$, which by the Yoneda Lemma is given by

$$\varphi_{X,Y}(f) = U(f) \circ \eta_X, \quad f \in \mathcal{D}(FX, Y), \tag{2.1}$$

where $\eta_X = \varphi_{X,FX}(\mathrm{id}_{FX}) : X \to UFX$. Clearly, the family of morphisms η_X is natural in X, giving rise to a natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \to UF$, called the *unit* of the adjunction (F, U, φ). Similarly, for a fixed object $Y \in \operatorname{Ob}\mathcal{D}$, the family of functions $\varphi^{-1}_{-,Y} = \{\varphi^{-1}_{X,Y}\}_{X \in \operatorname{Ob}\mathcal{C}}$ is a natural transformation from the representable functor $\mathcal{C}(-, UY) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ to the functor $\mathcal{D}(F(-), Y) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, which by the Yoneda Lemma is given by

$$\varphi^{-1}_{X,Y}(g) = \varepsilon_Y \circ F(g), \quad g \in \mathcal{C}(X, UY), \tag{2.2}$$

$$
\begin{tikzcd}
UTFX \arrow[r, "\eta_{UTFX}"] \arrow[d, "UTF(\eta_X)"'] \arrow[dr, equal] & UFUTFX \arrow[r, "U(e^T_{FUTFX})"] \arrow[d, "U(\varepsilon_{TFX})"] & UTFUTFX \arrow[d, "UT(\varepsilon_{TFX})"] \\
UTFUFX \arrow[r, "UT(\varepsilon_{FX})"] \arrow[d, "UTFU(e^T_{FX})"'] & UTFX \arrow[r, "U(e^T_{TFX})"] \arrow[d, "UT(e^T_{FX})"'] \arrow[dr, equal] & UTTFX \arrow[d, "U(m^T_{FX})"] \\
UTFUTFX \arrow[r, "UT(\varepsilon_{TFX})"] & UTTFX \arrow[r, "U(m^T_{FX})"] & UTFX
\end{tikzcd}
$$

Diagram 1: Proof of the identity axioms for the monad P.

where $\varepsilon_Y = \varphi^{-1}_{UY,Y}(\mathrm{id}_{UY}) : FUY \to Y$. The family of morphisms ε_Y is natural in Y, giving rise to a natural transformation $\varepsilon : FU \to \mathrm{Id}_{\mathcal{D}}$, called the *counit* of the adjunction (F, U, φ). The identities $\varphi^{-1}_{X,FX}(\eta_X) = \mathrm{id}_{FX}$ and $\varphi_{UY,Y}(\varepsilon_Y) = \mathrm{id}_{UY}$ translate into so called *triangular* or *counit-unit* equations:

$$\Big[FX \xrightarrow{F(\eta_X)} FUFX \xrightarrow{\varepsilon_{FX}} FX\Big] = \mathrm{id}_{FX}, \tag{2.3}$$

$$\Big[UY \xrightarrow{\eta_{UY}} UFUY \xrightarrow{\varepsilon_{U(\varepsilon_Y)}} UY\Big] = \mathrm{id}_{UY}\,. \tag{2.4}$$

Conversely, if $\eta : \mathrm{Id}_{\mathcal{C}} \to UF$ and $\varepsilon : FU \to \mathrm{Id}_{\mathcal{D}}$ are natural transformations satisfying equations (2.3) and (2.4), then the family of functions $\varphi_{X,Y} : \mathcal{D}(FX, Y) \to \mathcal{C}(X, UY)$ given by (2.1) is natural in X and Y, and each $\varphi_{X,Y}$ is invertible with the inverse given by (2.2). Therefore, an adjunction can equivalently be described as a quadruple $(F, U, \eta, \varepsilon)$, where $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$ are functors and $\eta : \mathrm{Id}_{\mathcal{C}} \to UF$ and $\varepsilon : FU \to \mathrm{Id}_{\mathcal{D}}$ are natural transformations subject to equations (2.3) and (2.4). We refer the interested reader to [3, Chapter IV, Theorem 2] for more equivalent definitions of an adjunction.

Every adjunction $(F, U, \eta, \varepsilon)$ from $\mathcal{C}$ to $\mathcal{D}$ gives rise to a monad (P, e^P, m^P) on $\mathcal{C}$, where $P = UF$, $e^P_X = \eta_X : X \to UFX$, and $m^P_X = U(\varepsilon_{FX}) : UFUFX \to UFX$. The following proposition is a mild generalization of this observation. The latter can be recovered by taking T to be the identity monad.

2.2 Proposition. *Let $(F, U, \eta, \varepsilon)$ be an adjunction from $\mathcal{C}$ to $\mathcal{D}$. Suppose that (T, e^T, m^T) is a monad on the category $\mathcal{D}$. Then the functor $P = UTF : \mathcal{C} \to \mathcal{C}$ equipped with the natural transformations*

$$e^P_X = \Big[X \xrightarrow{\eta_X} UFX \xrightarrow{U(e^T_{FX})} UTFX\Big], \tag{2.5}$$

$$m^P_X = \Big[UTFUTFX \xrightarrow{UT(\varepsilon_{TFX})} UTTFX \xrightarrow{U(m^T_{FX})} UTFX\Big] \tag{2.6}$$

is a monad on the category $\mathcal{C}$.

Proof. Let us check the monad axioms. The identity axioms are proven in Diagram 1. Each cell of this diagram commutes. The pair of top left triangles commute by the counit-unit equations (2.3) and (2.4). The pair of bottom right triangles commute by the identity axioms for the monad T. The commutativity of the top right square follows from the naturality of e^T, while the commutativity of the bottom left square follows from the naturality of ε. The associativity axiom coincides with the exterior of Diagram 2. The commutativity of the two squares on the left follows from the naturality of ε. The top right square commutes by the naturality of m^T. Finally, the bottom right square commutes by the associativity axiom for the monad T. ☐

2.3 Remark. Proposition 2.2 allows us to translate a monad on the category $\mathcal{D}$ into a monad on the category $\mathcal{C}$. In functional programming and denotational semantics, we are primarily interested in monads that are strong [4, Definition 3.2]. We recall that a monad (T, e^T, m^T) on a cartesian

$$\begin{tikzcd}
UTFUTFUTFX \arrow[r, "UT(\varepsilon_{TFUTFX})"] \arrow[d, "UTFUT(\varepsilon_{TFX})"'] & UTTFUTFX \arrow[r, "U(m^T_{FUTFX})"] \arrow[d, "UTT(\varepsilon_{TFX})"] & UTFUTFX \arrow[d, "UT(\varepsilon_{TFX})"] \\
UTFUTTFX \arrow[r, "UT(\varepsilon_{TTFX})"] \arrow[d, "UTFU(m^T_{FX})"'] & UTTTFX \arrow[r, "U(m^T_{TFX})"] \arrow[d, "UT(m^T_{FX})"] & UTTFX \arrow[d, "U(m^T_{FX})"] \\
UTFUTFX \arrow[r, "UT(\varepsilon_{TFX})"] & UTTFX \arrow[r, "U(m^T_{FX})"] & UTFX
\end{tikzcd}$$

Diagram 2: Proof of the associativity axiom for the monad P.

category $\mathcal{C}$ is *strong* if it is equipped with a transformation $t_{X,Y} : TX \times Y \to T(X \times Y)$ natural in X and Y and compatible with both the cartesian and monad structures; see [4, Definition 3.2] for the precise compatibility conditions. Under what conditions is the translation of a strong monad along an adjunction also a strong monad? I don't have a good answer. In each of the examples we consider below, this question can be resolved in an ad hoc manner. However, I am not aware of a general condition that applies to all the examples. That is why I am going to ignore this issue henceforth. Note that in two out of the four examples we will need the assumption that the monad being translated is strong.

3 State monad transformer

For the sake of simplicity, we assume that the categories $\mathcal{C}$ and $\mathcal{D}$ are the category **Set** of sets. An arbitrary set S gives rise to an adjunction $(F, U, \eta, \varepsilon)$, where $F = - \times S$, $U = (-)^S$, $\eta_X : X \to (X \times S)^S$ is given by $\eta_X(x) = \lambda s.(x, s)$, and $\varepsilon_X : X^S \times S \to X$ is given by $\varepsilon_X(f, s) = f(s)$. The monad (P, e^P, m^P) associated with this adjunction is the state monad with the state S: $PX = (X \times S)^S$, $e^P_X(x) = \eta_X(x) = \lambda s.(x, s)$, and $m^P_X(g) = \varepsilon_{FX} \circ g = \lambda s.\ \mathbf{let}\ (f, s') = g(s)\ \mathbf{in}\ f(s')$.

More generally, suppose (T, e^T, m^T) is a monad on **Set**. Let us compute explicitly the monad (P, e^P, m^P) obtained by translating T along the adjunction $(F, U, \eta, \varepsilon)$. We have: $PX = (T(X \times S))^S$, $e^P_X(x) = e^T_{FX} \circ \eta_X(x) = \lambda s.e^T_{X \times S}(x, s)$, $m^P_X(g) = m^T_{FX} \circ T(\varepsilon_{TFX}) \circ g$. Translating the last equation into Haskell notation, we obtain:

```
join :: Monad m => (s -> m (s -> m (a, s), s)) -> s -> m (a, s)
join g = join . fmap ev . g where ev (f, s’) = f s’
```

which can be transformed as follows:

```
join g = \s -> join $ fmap ev $ g s

       -- definitions of ‘fmap’ and ‘join’
       = \s -> (g s >>= (return . ev)) >>= id

       -- associativity axiom for monads
       = \s -> g s >>= (\(f, s’) -> return (ev (f, s’)) >>= id)

       -- left identity axiom for monads
       = \s -> g s >>= (\(f, s’) -> f s’)

       -- syntactic sugar
       = \s -> do (f, s’) <- g s; f s’
```

Modulo some newtype constructor wrapping/unwrapping this is precisely the multiplication in the monad `StateT s m`.

4 Writer monad transformer

Let $\mathcal{C}$ be the category **Set** of sets. Let M be a monoid with the binary operation $\cdot : M \times M \to M$, $(m_1, m_2) \mapsto m_1 \cdot m_2$, and the neutral element $1 \in M$. Let $\mathcal{D}$ be the category M-**Set** of M-sets: objects are M-sets, i.e., pairs (X, a), where X is a set and $a : X \times M \to X$, $(x, m) \mapsto x^m$ is an action morphism such that $x^1 = x$ and $(x^{m_1})^{m_2} = x^{m_1 \cdot m_2}$, for all $x \in X$ and $m_1, m_2 \in M$, and morphisms are M-equivariant maps, i.e., a morphism $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that $f(x^m) = f(x)^m$, for all $x \in X$ and $m \in M$. There is an adjunction $(F, U, \eta, \varepsilon)$ from **Set** to M-**Set**. The functor $F : \mathbf{Set} \to M\text{-}\mathbf{Set}$ maps a set X to the free M-set $X \times M$ with the action $(X \times M) \times M \to X \times M$ given by $(x, m)^n = (x, m \cdot n)$. The functor $U : M\text{-}\mathbf{Set} \to \mathbf{Set}$ is the forgetful functor that maps an M-set (X, a) to its underlying set X. The unit $\eta_X : X \to X \times M$ is given by $\eta_X(x) = (x, 1)$, and the counit $\varepsilon_{(X,a)} : X \times M \to X$ is simply the action morphism a. The monad associated with this adjunction is the writer monad with the monoid M.

We conclude by Proposition 2.2 that a monad (T, e^T, m^T) on the category $\mathcal{D} = M$-**Set** gives rise to a monad on $\mathcal{C} = \mathbf{Set}$. However, this is not what we would like to have: we would like to produce a monad on **Set** out of another monad (T, e^T, m^T) on **Set**, not on M-**Set**. This is possible if the monad T is strong. Let $t_{X,Y} : TX \times Y \to T(X \times Y)$ be the strength of the monad T. Then (T, e^T, m^T) induces a monad $(\bar{T}, e^{\bar{T}}, m^{\bar{T}})$ on the category M-**Set**. Namely, if (X, a) is an M-set, then the set TX becomes an M-set if we equip it with the action

$$b = \Big[TX \times M \xrightarrow{t_{X,M}} T(X \times M) \xrightarrow{T(a)} TX\Big]. \tag{4.1}$$

Let us prove that b is an action, i.e., that it satisfies the identity and associativity conditions. It is convenient to first express these conditions diagrammatically. A map $a : X \times M \to M$ is an action if it satisfies the identity axiom:

$$\Big[X \xrightarrow[\sim]{\rho_X} X \times \mathbb{1} \xrightarrow{\mathrm{id}_X \times 1_M} X \times M \xrightarrow{a} X\Big] = \mathrm{id}_X,$$

and the associativity axiom:

$$\begin{tikzcd}
(X \times M) \times M \arrow[r, "a \times \mathrm{id}_M"] \arrow[d, "\alpha_{X,M,M}"', "\wr"] & X \times M \arrow[dd, "a"] \\
X \times (M \times M) \arrow[d, "\mathrm{id}_X \times \cdot"'] & \\
X \times M \arrow[r, "a"] & X
\end{tikzcd}$$

Here ρ and α are the right unit and associativity constraints of the monoidal structure induced by the cartesian product, $\mathbb{1} = \{*\}$ is the terminal object (a singleton), and $1_M : \mathbb{1} \to M$, $* \mapsto 1$. Suppose that $a : X \times M \to X$ is an action. Let us prove that the map b given by (4.1) is also an action. The identity axiom is proven in the following diagram:

$$\begin{tikzcd}
TX \times \mathbb{1} \arrow[r, "\mathrm{id}_{TX} \times 1_M"] \arrow[d, "t_{X,\mathbb{1}}"] & TX \times M \arrow[d, "t_{X,M}"'] \arrow[dd, bend left=60, "b"] \\
T(X \times \mathbb{1}) \arrow[r, "T(\mathrm{id}_X \times 1_M)"] & T(X \times M) \arrow[d, "T(a)"'] \\
TX \arrow[u, "T(\rho_X)"'] \arrow[uu, bend left=60, "\rho_{TX}"] \arrow[r, equal] & TX
\end{tikzcd}$$

The top square commutes by the naturality of the strength t. The bottom square commutes by the identity axiom for the action a (and functoriality of T). The left triangle is one of the strength axioms, and the right triangle is the definition of b. The associativity axiom coincides with the exterior of the following diagram:

$$\begin{tikzcd}
(TX \times M) \times M \arrow[r, "t_{X,M} \times \mathrm{id}_M"] \arrow[dd, "\alpha_{TX,M,M}"'] & T(X \times M) \times M \arrow[r, "T(a) \times \mathrm{id}_M"] \arrow[d, "t_{X \times M,M}"] & TX \times M \arrow[d, "t_{X,M}"] \\
 & T((X \times M) \times M) \arrow[r, "T(a \times \mathrm{id}_M)"] \arrow[d, "T(\alpha_{X,M,M})"] & T(X \times M) \arrow[dd, "T(a)"] \\
TX \times (M \times M) \arrow[r, "t_{X,M \times M}"] \arrow[d, "\mathrm{id}_{TX} \times \cdot"] & T(X \times (M \times M)) \arrow[d, "T(\mathrm{id}_X \times \cdot)"] & \\
TX \times M \arrow[r, "t_{X,M}"] & T(X \times M) \arrow[r, "T(a)"] & TX
\end{tikzcd}$$

The left pentagon is one of the strength axioms. The right pentagon commutes by the associativity condition for a. The remaining squares commute by the naturality of t.

We have proven that once (X, a) is an M-set, the pair (TX, b), where b is given by (4.1), is also an M-set. We set $\bar{T}(X, a) = (TX, b)$. Let us check that if $f : (X, a) \to (X', a')$ is an M-equivariant map, then $T(f) : TX \to TX'$ is actually an M-equivariant map $\bar{T}(X, a) \to \bar{T}(X', a')$. This is easy: the equivariance of f is expressed by the commutativity of the diagram

$$\begin{tikzcd}
X \times M \arrow[r, "f \times \mathrm{id}_M"] \arrow[d, "a"'] & X' \times M \arrow[d, "a'"] \\
X \arrow[r, "f"] & X'
\end{tikzcd}$$

The naturality of t and the functoriality of T imply that the following diagram commutes, too:

$$\begin{tikzcd}
TX \times M \arrow[r, "Tf \times \mathrm{id}_M"] \arrow[d, "t_{X,M}"'] & TX' \times M \arrow[d, "t_{X',M}"] \\
T(X \times M) \arrow[r, "T(f \times \mathrm{id}_M)"] \arrow[d, "T(a)"'] & T(X' \times M) \arrow[d, "T(a')"] \\
TX \arrow[r, "T(f)"] & TX'
\end{tikzcd}$$

The vertical compositions are precisely the action morphisms of $\bar{T}(X, a)$ and $\bar{T}(X', a')$. Therefore, the above diagram expresses the fact that the map $T(f) : TX \to TX'$ is indeed an M-equivariant map $\bar{T}(X, a) \to \bar{T}(X', a')$. Hence, we can set $\bar{T}(f) = T(f)$. Then $\bar{T}$ is a functor from the category M-**Set** to itself. The functoriality of $\bar{T}$ follows immediately from that of T.

One can also prove that the natural transformations $e^T_X : X \to TX$ and $m^T_X : TTX \to TX$ induce natural transformations $e^{\bar{T}}_{(X,a)} : (X, a) \to \bar{T}(X, a)$ and $m^{\bar{T}}_{(X,a)} : \bar{T}\bar{T}(X, a) \to \bar{T}(X, a)$. It suffices to check that e^T_X and m^T_X are M-equivariant if X is an M-set, which follows directly from the strength axioms expressing the compatibility of t with the unit and multiplication of the monad T. For example,

the equivariance of m^T_X is proven in the following diagram:

$$\begin{tikzcd}
TTX \times M \arrow[r, "m^T_X \times \mathrm{id}_M"] \arrow[d, "t_{TX,M}"'] & TX \times M \arrow[dd, "t_{X,M}"] \\
T(TX \times M) \arrow[d, "T(t_{X,M})"'] & \\
TT(X \times M) \arrow[r, "m^T_{X\times M}"] \arrow[d, "TT(a)"'] & T(X \times M) \arrow[d, "T(a)"] \\
TTX \arrow[r, "m^T_X"] & TX
\end{tikzcd}$$

The pentagon is a strength axiom, and the square is a consequence of the naturality of m^T. A similar argument shows that e^T_X is also M-equivariant.

It is straightforward that the natural transformations $e^{\bar{T}}_{(X,a)}$ and $m^{\bar{T}}_{(X,a)}$ satisfy the monad laws, because the underlying maps e^T_X and m^T_X satisfy these laws, and because the action of $\bar{T}$ on morphisms coincides with that of T.

Thus we have shown that a strong monad (T, e^T, m^T) on the category **Set** induces a monad $(\bar{T}, e^{\bar{T}}, m^{\bar{T}})$ on the category M- **Set**, which can now be translated along the adjunction $(F, U, \eta, \varepsilon)$ by Proposition 2.2. Let us compute the obtained monad (P, e^P, m^P) explicitly. The functor $P = U\bar{T}F : \mathbf{Set} \to \mathbf{Set}$ is given by $PX = T(X \times M)$. The unit $e^P_X : X \to T(X \times M)$ of the monad P is given by $e^P_X = e^T_{X\times M} \circ \eta_X = \lambda x.\ e^T_{X\times M}(x, 1)$, which is the `return` method of the writer monad transformer. Let us compute the multiplication. By Proposition 2.2, $m^P_X = m^T_{X\times M} \circ T(b)$, where $b = T(a) \circ t_{X\times M} : T(X \times M) \times M \to T(X \times M)$ is the action morphisms of $T(X \times M)$, and $a : (X \times M) \times M \to X \times M$, $((x, m_1), m_2) \mapsto (x, m_1 \cdot m_2)$ is the action morphism of $X \times M$. Translating this into Haskell notation, we obtain:

```
join = join . fmap b
    where
      b                = fmap a . strength
      a ((x, m1), m2) = (x, m1 ‘mappend‘ m2)
      strength (c, m) = c >>= (\r -> return (r, m))
```

Eta-expanding this definition, we obtain:

```
join z = join (fmap b z)

       -- definitions of ‘join’ and ‘fmap’
       = (z >>= (return . b)) >>= id

       -- associativity axiom for monads
       = z >>= \p -> return (b p) >>= id

       -- left identity axiom for monads
       = z >>= \p -> b p

       -- definition of ‘b’
       = z >>= \p -> fmap a (strength p)

       -- definition of ‘fmap’
       = z >>= \(c, m) -> strength (c, m) >>= (return . a)
```

The expression `strength (c, m) >>= (return . a)` can be further transformed as follows:

```
strength (c, m) >>= (return . a)

    -- definition of ‘strength’
    = (c >>= (\r -> return (r, m))) >>= (return . a)

    -- associativity axiom for monads
    = c >>= \(x, n) -> return ((x, n), m) >>= (return . a)

    -- left identity axiom for monads
    = c >>= \(x, n) -> return $ a ((x, n), m)

    -- definition of ‘a’
    = c >>= \(x, n) -> return (x, n ‘mappend‘ m)
```

Therefore

```
join z = z >>= \(c, m) -> c >>= \(x, n) -> return (x, n ‘mappend‘ m)

      -- syntactic sugar
      = do (c, m) <- z
           (x, n) <- c
           return (x, n ‘mappend‘ m)
```

which is the multiplication in the monad `WriterT w m`, modulo some newtype contructor wrapping/unwrapping.

4.1 Remark. There are other ways to explain why the functor $W = - \times M$ is part of a monad when M is a monoid. Let $\mathcal{C}$ be a cartesian category. By currying the product functor $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, we obtain a functor $\Gamma : \mathcal{C} \to [\mathcal{C}, \mathcal{C}]$, where $[\mathcal{C}, \mathcal{C}]$ denotes the category of functors from $\mathcal{C}$ to itself. The functor Γ maps an object $X \in \operatorname{Ob} \mathcal{C}$ to the functor $- \times X$ and a morphism f to $f \times \mathrm{id}_X$. The categories $\mathcal{C}$ and $[\mathcal{C}, \mathcal{C}]$ are naturally monoidal: in the former, the monoidal structure is given by cartesian product, and in the latter it is given by functor composition. The functor Γ is monoidal: $\Gamma(X \times Y) = - \times (X \times Y) \simeq (- \times Y) \circ (- \times X) = \Gamma Y \circ \Gamma X$. The natural isomorphism is given by the associativity constraint of the monoidal structure induced by cartesian product. Therefore, Γ takes monoids in $\mathcal{C}$ to monoids in $[\mathcal{C}, \mathcal{C}]$. The latter are precisely monads on the category $\mathcal{C}$.

4.2 Remark. The category $\mathcal{D} = M\text{-}\mathbf{Set}$ introduced in this section is precisely the category of algebras over the monad $W = - \times M$, and the adjunction from $\mathcal{C}$ to $\mathcal{D}$ constructed above is an instance of the general construction of an adjunction from the category $\mathcal{C}$ to the category $\mathcal{C}^W$ of algebras over W.

4.3 Remark. Let T be a strong monad on $\mathcal{C}$. Suppose that M is a monoid in $\mathcal{C}$. The strength $t_{X,M} : TX \times M \to T(X \times M)$ is a natural transformation $t : (- \times M) \circ T \to T \circ (- \times M)$. The functor $W = - \times M$ is a monad, and t can be viewed as a *distributive law* of the monad T over W. In fact, the four axioms of distributive laws reduce in this case to the four axioms of strengths. This yields an alternative argument for why the composition of the monads W and T is again a monad. Furthermore, this also explains why the monad T lifts to a monad $\bar{T}$ on the category $\mathcal{C}^W$ of algebras over the monad W: distributive laws $WT \to TW$ are in bijection with such liftings [1].

5 Reader monad transformer

Although the considerations below can be carried out in any cartesian closed category $\mathcal{C}$, we assume for the sake of simplicity that $\mathcal{C}$ is the category $\mathbf{Set}$ of sets. Let E be a set. Define $\mathcal{D}$ to be the category whose objects are the objects of $\mathcal{C}$, and for each pair of objects X and Y, the set of morphisms $\mathcal{D}(X, Y)$ is equal to $\mathcal{C}(X \times E, Y)$. We can think of $\mathcal{D}(X, Y)$ as the set of families of functions from X to Y

parametrized by E. The identity morphism of an object X in $\mathcal{D}$ is the projection $\mathrm{pr}_1 : X \times E \to X$. Composition of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{D}$ (i.e., of maps $f : X \times E \to Y$ and $g : Y \times E \to Z$ in $\mathcal{C}$) is given by

$$g * f = \Big[X \times E \xrightarrow{\mathrm{id}_X \times \Delta} X \times (E \times E) \xrightarrow{\alpha^{-1}_{X,E,E}} (X \times E) \times E \xrightarrow{f \times \mathrm{id}_E} Y \times E \xrightarrow{g} X\Big].$$

Here $\Delta : E \to E \times E$ is the diagonal map. In other words, $(g * f)(x, e) = g(f(x, e), e)$ for all $(x, e) \in X \times E$, but it is helpful to have a diagrammatic representation of composition that does not refer to elements.

There is an adjunction $(F, U, \eta, \varepsilon)$ from $\mathcal{C}$ to $\mathcal{D}$. The functor $F : \mathcal{C} \to \mathcal{D}$ maps each object $X \in \mathrm{Ob}\,\mathcal{C}$ to itself and each morphism $f : X \to Y$ to the composition $f \circ \mathrm{pr}_1 : X \times E \to Y$. In other words, to a function f the functor F assigns the constant family of functions. With this interpretation, F is clearly a functor. The functor $U : \mathcal{D} \to \mathcal{C}$ maps an object $X \in \mathrm{Ob}\,\mathcal{D} = \mathrm{Ob}\,\mathcal{C}$ to the exponential X^E, and a morphism $f : X \to Y$ in $\mathcal{D}$ (i.e., a morphism $f : X \times E \to Y$ in $\mathcal{C}$) to the morphisms corresponding to the composite

$$f * \mathrm{ev} = \Big[X^E \times E \xrightarrow{\mathrm{id}_{X^E} \times \Delta} X^E \times (E \times E) \xrightarrow{\alpha^{-1}_{X^E,E,E}} (X^E \times E) \times E \xrightarrow{\mathrm{ev} \times \mathrm{id}_E} X \times E \xrightarrow{f} Y\Big]$$

by the closedness of the category $\mathcal{C}$. Here $\mathrm{ev} : X^E \times E \to X$ is the evaluation morphism. Because we are assuming that $\mathcal{C}$ is the category of sets, $U(f)$ can be written as $\lambda g.\ \lambda e.\ f(g(e), e)$. Informally, an E-indexed family of functions $\{f_e : X \to Y\}_{e \in E}$ is mapped to the function that takes an E-indexed family of elements $\{x_e\}_{e \in E}$ of the set X as input and applies each function to the corresponding element, producing a new E-indexed family of elements $\{f_e(x_e)\}_{e \in E}$ of elements of the set Y. This makes it obvious that U is a functor. The unit $\eta_X : X \to FUX = X^E$ is given by $\eta_X(x) = \lambda e.\ x$. The counit $\varepsilon_X : FUX \to X$ is a morphism in $\mathcal{D}$ represented by the evaluation morphism $\mathrm{ev} : X^E \times E \to X$ in $\mathcal{C}$. The monad associated with this adjunction is precisely the reader monad with the environment E.

Let (T, e^T, m^T, t) be a strong monad on the category $\mathcal{C}$. Like in the case of writer monad transformer, we would like to lift T to a monad $\bar{T}$ on the category $\mathcal{D}$, which we then could translate back to $\mathcal{C}$ along the adjunction $(F, U, \eta, \varepsilon)$. The functor T is lifted to the category $\mathcal{D}$ as follows: $\bar{T}X = TX$ and for each morphism $f \in \mathcal{D}(X, Y) = \mathcal{C}(X \times E, Y)$, we set

$$\bar{T}(f) = \Big[TX \times E \xrightarrow{t_{X,E}} T(X \times E) \xrightarrow{T(f)} TY\Big].$$

Let us check that $\bar{T}$ preserves composition and identities. Let $f \in \mathcal{D}(X, Y) = \mathcal{C}(X \times E, Y)$ and $g \in \mathcal{D}(Y, Z) = \mathcal{C}(Y \times E, Z)$. The equation $\bar{T}(g * f) = \bar{T}(g) * \bar{T}(f)$ coincides with the exterior of the diagram:

$$\begin{tikzcd}
TX \times E \arrow[r, "t_{X,E}"] \arrow[d, "\mathrm{id}_{TX} \times \Delta"'] & T(X \times E) \arrow[d, "T(\mathrm{id}_X \times \Delta)"] & \\
TX \times (E \times E) \arrow[r, "t_{X,E \times E}"] \arrow[d, "\alpha^{-1}_{TX,E,E}"'] & T(X \times (E \times E)) \arrow[dd, "T(\alpha^{-1}_{X,E,E})"] & \\
(TX \times E) \times E \arrow[d, "t_{X,E} \times \mathrm{id}_E"'] & & \\
T(X \times E) \times E \arrow[r, "t_{X \times E,E}"] \arrow[d, "T(f) \times \mathrm{id}_E"'] & T((X \times E) \times E) \arrow[d, "T(f \times \mathrm{id}_E)"] & \\
TY \times E \arrow[r, "t_{Y,E}"] & T(Y \times E) \arrow[r, "T(g)"] & TZ
\end{tikzcd}$$

The squares commute by the naturality of t. The pentagon is, up to the orientation of the associativity isomorphism, one of the strength axioms. Preservation of identities follows from the diagram:

$$\begin{tikzcd}
TX \times E \arrow[r, "t_{X,E}"] \arrow[d, "\mathrm{id}_{TX} \times !_E"'] & T(X \times E) \arrow[d, "T(\mathrm{id}_X \times !_E)"'] \arrow[dr, "T(\mathrm{pr}_1)"] & \\
TX \times \mathbb{1} \arrow[r, "t_{X,\mathbb{1}}"] \arrow[rr, bend right, "\mathrm{pr}_1"'] & T(X \times \mathbb{1}) \arrow[r, "T(\mathrm{pr}_1)"] & TX
\end{tikzcd}$$

The square commutes by the naturality of t. The commutativity of the right triangle follows from the obvious identity $\mathrm{pr}_1 \circ (\mathrm{id}_X \times !_E) = \mathrm{pr}_1$ and functoriality of T. The bottom triangle is one of the strength axioms (note that $\rho_X^{-1} = \mathrm{pr}_1 : X \times \mathbb{1} \to X$). The left-bottom composite is equal to $\mathrm{pr}_1 : TX \times E \to TX$, which represents the identity morphism of TX in the category $\mathcal{D}$.

We have shown that $\bar{T}$ is a functor from the category $\mathcal{D}$ to itself. Let us check that the families of morphisms

$$\begin{aligned}
e_X^{\bar{T}} &= e_X^T \circ \mathrm{pr}_1 = F(e_X^T) \in \mathcal{C}(X \times E, TX) = \mathcal{D}(X, \bar{T}X), \\
m_X^{\bar{T}} &= m_X^T \circ \mathrm{pr}_1 = F(m_X^T) \in \mathcal{C}(TTX \times E, TX) = \mathcal{D}(\bar{T}\bar{T}X, \bar{T}X)
\end{aligned}$$

are natural transformations $\mathrm{Id}_{\mathcal{D}} \to \bar{T}$ and $\bar{T}\bar{T} \to \bar{T}$. We only give a proof for the second family. Let $f \in \mathcal{D}(X, Y) = \mathcal{C}(X \times E, Y)$. We have to show that the diagram

$$\begin{tikzcd}
\bar{T}\bar{T}X \arrow[r, "\bar{T}\bar{T}(f)"] \arrow[d, "m_X^{\bar{T}}"'] & \bar{T}\bar{T}Y \arrow[d, "m_Y^{\bar{T}}"] \\
\bar{T}X \arrow[r, "\bar{T}(f)"] & \bar{T}Y
\end{tikzcd}$$

commutes in $\mathcal{D}$. Using the definition of composition in $\mathcal{D}$ and the definition of $\bar{T}$, it is not difficult to convince yourself that this diagram coincides with the exterior of the following diagram:

$$\begin{tikzcd}
TTX \times E \arrow[d, "\mathrm{id}_{TTX} \times \Delta"'] & & & \\
TTX \times (E \times E) \arrow[d, "\alpha^{-1}_{TTX,E,E}"'] & & & \\
(TTX \times E) \times E \arrow[r, "t_{TX,E} \times \mathrm{id}_E"] \arrow[d, "\mathrm{pr}_1 \times \mathrm{id}_E"'] & T(TX \times E) \times E \arrow[r, "T(t_{X,E}) \times \mathrm{id}_E"] \arrow[d, "\mathrm{pr}_1"'] & TT(X \times E) \times E \arrow[r, "TT(f) \times \mathrm{id}_E"] \arrow[d, "\mathrm{pr}_1"'] & TTY \times E \arrow[d, "\mathrm{pr}_1"] \\
TTX \times E \arrow[r, "t_{TX,E}"] \arrow[d, "m_X^T \times \mathrm{id}_E"'] & T(TX \times E) \arrow[r, "T(t_{X,E})"] & TT(X \times E) \arrow[r, "TT(f)"] \arrow[d, "m_{X \times E}^T"'] & TTY \arrow[d, "t_Y^T"] \\
TX \times E \arrow[rr, "t_{X,E}"] & & T(X \times E) \arrow[r, "T(f)"] & TY
\end{tikzcd}$$

The three top squares commute for trivial reasons. The remaining square commutes by the naturality of m^T. The pentagon is one of the strength axioms.

We have shown that $e_X^{\bar{T}} = F(e_X^T)$ and $m_X^{\bar{T}} = F(m_X^T)$ give rise to natural transformations $\mathrm{Id}_{\mathcal{D}} \to \bar{T}$ and $\bar{T}\bar{T} \to \bar{T}$, respectively. That they satisfy the monad laws follows from the fact that e^T and m^T satisfy the monad laws, and from the functoriality of F.

Thus, we have lifted the monad (T, e^T, m^T) on the category $\mathcal{C}$ to the monad $(\bar{T}, e^{\bar{T}}, m^{\bar{T}})$ on the category $\mathcal{D}$, which we can now translate along the adjunction $(F, U, \eta, \varepsilon)$ back to the category $\mathcal{C}$. The composite monad (P, e^P, m^P) is given by $PX = (TX)^E$,

$$e_X^P(x) = (\lambda g.\ \lambda e.\ e_X^T(g(e)))(\lambda e.\ x) = \lambda e. e_X^T(x),$$
$$m_X^P = (\lambda g.\ \lambda e.\ m_X^T(g(e))) \circ (\lambda h.\ \lambda e.\ T(\mathrm{ev})(t_{(TX)^E,E}(h(e), e))),$$

where $\mathrm{ev} : (TX)^E \times E \to TX$ is the evaluation map. Eta-expanding the definition of m_X^P, we obtain:

$$m_X^P(h) = \lambda e.\ m_X^T(T(\mathrm{ev})(t_{(TX)^E,E}(h(e), e))).$$

Translating this into Haskell notation, we obtain:

```
join h = \e -> join $ fmap ev $ strength (h e, e)
    where
      ev (f, v)       = f v
      strength (c, y) = c >>= \x -> return (x, y)
```

which can be transformed as follows:

```
join h = \e -> join $ fmap ev $ strength (h e, e)

       -- definitions of ‘join’ and ‘fmap’
       = \e -> (strength (h e, e) >>= (return . ev)) >>= id

       -- associativity axiom for monads
       = \e -> strength (h e, e) >>= \(f, v) -> return (ev (f, v)) >>= id

       -- left identity axiom for monads
       = \e -> strength (h e, e) >>= \(f, v) -> ev (f, v)

       -- definition of ‘ev’
       = \e -> strength (h e, e) >>= \(f, v) -> f v

       -- definition of ‘strength’
       = \e -> (h e >>= \x -> return (x, e)) >>= \(f, v) -> f v

       -- associativity axiom for monads
       = \e -> h e >>= \x -> return (x, e) >>= \(f, v) -> f v

       -- left identity axiom for monads
       = \e -> h e >>= \f -> f e

       -- syntactic sugar
       = \e -> do f <- h e
                  f e
```

Modulo newtype constructor wrapping/unwrapping, this is precisely the multiplication in the monad `ReaderT e m`.

5.1 Remark. Here is another, higher-level explanation of why the functor $R = (-)^E$ is part of a monad. Let $\mathcal{C}$ be a cartesian closed category. Currying the internal hom functor $(-)^- : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, $(X, Y) \mapsto Y^X$, we obtain a functor $\Gamma : \mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathcal{C}]$, $X \mapsto (-)^X$. The functor Γ is monoidal: $\Gamma(X \times Y) = (-)^{X \times Y} \simeq (-)^X \circ (-)^Y = \Gamma X \circ \Gamma Y$. Any object E of the category $\mathcal{C}$ is naturally a coalgebra (more appropriately called ‘comonoid’) in $\mathcal{C}$: the comultiplication $\Delta : E \to E \times E$ is the diagonal morphism (a unique morphism such that $\mathrm{pr}_1 \circ \Delta = \mathrm{pr}_2 \circ \Delta = \mathrm{id}_E$), and the counit $\varepsilon : E \to \mathbb{1}$

is the final morphism $!_E$. Therefore, E is a monoid in the opposite category $\mathcal{C}^{\mathrm{op}}$. The monoidal functor Γ takes monoids in $\mathcal{C}^{\mathrm{op}}$ to monoids in $[\mathcal{C}, \mathcal{C}]$. The latter are precisely monads on the category $\mathcal{C}$. Therefore, ΓE has the structure of a monad.

5.2 Remark. The category $\mathcal{D}$ introduced at the beginning of this section is isomorphic to the Kleisli category $\mathcal{C}_R$ of the monad R: $\mathrm{Ob}\,\mathcal{C}_R = \mathrm{Ob}\,\mathcal{C}$, $\mathcal{C}_R(X, Y) = \mathcal{C}(X, RY) = \mathcal{C}(X, Y^E)$, the identity of an object X is the unit $e^R_X : X \to X^E$ of the monad, and composition is given by $g \circ_R f = m^R_Z \circ R(g) \circ f$, for all $f \in \mathcal{C}_R(X, Y) = \mathcal{C}(X, Y^E)$, $g \in \mathcal{C}_R(Y, Z) = \mathcal{C}(Y, Z^E)$. The isomorphism between $\mathcal{D}$ and $\mathcal{C}_R$ is given by the identity map on objects and the closedness bijection $\mathcal{C}(X \times E, Y) \simeq \mathcal{C}(X, Y^E)$ on morphisms. The adjunction from $\mathcal{C}$ to $\mathcal{D}$ introduced above is an instance of a general construction, the adjunction from the category $\mathcal{C}$ to the Kleisli category of the monad R, translated along this isomorphism of categories.

6 Error monad transformer

Let $\mathcal{C}$ be the category of sets. Let E be a set. Consider the category $\mathcal{D} = E/\mathcal{C}$, the under category (also known as coslice category) of the object E: objects are maps $\varphi : E \to X$ in $\mathcal{C}$, and a morphism from $\varphi : E \to X$ to $\psi : E \to Y$ is a map $f : X \to Y$ such that $f \circ \varphi = \psi$. Composition and identities are the obvious ones. There is an adjunction $(F, U, \eta, \varepsilon)$ from $\mathcal{C}$ to $\mathcal{D}$. The functor F maps a set X to the morphism $\mathrm{inl} : E \to E + X$ and a map f to $\mathrm{id}_E + f$. Here $+$ denotes disjoint union of sets. The functor $U : \mathcal{D} \to \mathcal{C}$ is the forgetful functor, mapping a function $E \to X$ to its codomain X. The unit $\eta_X : X \to E + X$ is the map inr, and the counit is the morphism $\varepsilon_{\varphi : E \to X} : (E \xrightarrow{\mathrm{inl}} E + X) \to (E \xrightarrow{\varphi} X)$ given by $\varphi \vee \mathrm{id}_X$. Here for a pair of maps $f : X \to Z$ and $g : Y \to Z$, $f \vee g$ denotes the unique map $X + Y \to Z$ such that $(f \vee g) \circ \mathrm{inl} = f$ and $(f \vee g) \circ \mathrm{inr} = g$. The monad associated with this adjunction is precisely the error monad with the set of errors E.

Let (T, e^T, m^T) be a monad on the category $\mathcal{C} = \mathbf{Set}$. First of all, the functor T lifts to a functor $\bar{T}$ on the category $\mathcal{D} = E/\mathbf{Set}$: $\bar{T}(\varphi : E \to X) = T(\varphi) \circ e^T_E$, and $\bar{T}(f) = T(f)$ for each $f \in \mathcal{D}(\varphi : E \to X, \psi : E \to Y) \subset \mathcal{C}(X, Y)$. Clearly, $\bar{T}$ preserves identities and composition. Furthermore, the families of morphisms $e^T_X : X \to TX$ and $m^T_X : TTX \to TX$ can also be viewed as natural transformations $e^{\bar{T}}_{\varphi : E \to X} : (E \xrightarrow{\varphi} X) \to (E \xrightarrow{T(\varphi) \circ e^T_E} TX)$ and

$$m^{\bar{T}}_{\varphi : E \to X} : (E \xrightarrow{TT(\varphi) \circ T(e^T_E) \circ e^T_E} TTX) \to (E \xrightarrow{T(\varphi) \circ e^T_E} TX).$$

That $e^{\bar{T}}$ is a morphism in the category $\mathcal{D}$ is a consequence of the naturality of e^T, and that $m^{\bar{T}}$ is a morphism in $\mathcal{D}$ follows from the naturality of m^T and the right identity axiom for monads. The naturality of $e^{\bar{T}}$ and $m^{\bar{T}}$ follows from the naturality of e^T and m^T. Hence, $(\bar{T}, e^{\bar{T}}, m^{\bar{T}})$ is a monad on the category $\mathcal{D}$, which we can now translate back to the category $\mathcal{C}$. The composite monad (P, e^P, m^P) is given by $PX = T(E + X)$, $e^P_X = e^T_{E+X} \circ \mathrm{inr}$, $m^P_X = m^T_{E+X} \circ T((T(\mathrm{inl}) \circ e^T_E) \vee \mathrm{id}_{T(E+X)})$. Translating the last equation into Haskell notation, we obtain:

```
join = join . fmap eps
    where
      eps = either (fmap Left . return) id
```

Eta-expanding and transforming this equation, we obtain:

```
join z = join (fmap eps z)

       -- definitions of ‘join’ and ‘fmap’
       = (z >>= (return . eps)) >>= id

       -- axioms of monads (see above)
       = z >>= eps
```

```
-- syntactic sugar
= do y <- z
     eps y

-- definition of ‘either’
= do y <- z
     case y of
       Left e  -> (fmap Left . return) e
       Right x -> id x

-- naturality of ‘return’: fmap Left . return = return . Left
= do y <- z
     case y of
       Left e  -> return (Left e)
       Right x -> x
```

Modulo newtype constructor wrapping/unwrapping this is precisely the multiplication in `ErrorT e m`.

References

[1] Jon Beck. Distributive laws. In *Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67)*, pages 119–140. Springer, Berlin, 1969.

[2] Derek Elkins. Calculating monads with category theory. *The Monad.Reader*, 13, 2009.

[3] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[4] Eugenio Moggi. Notions of computation and monads. *Inform. and Comput.*, 93(1):55–92, 1991. Selections from the 1989 IEEE Symposium on Logic in Computer Science.